FUNDAMENTAL THEOREM OF ALGEBRA

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1. Introduction

The fundamental theorem of algebra, which states that every nonconstant polynomial has a root, has posed an annoying pedagogical problem. Although the result is easy enough to teach to high schoolers, any proof is waived off. Even in college math classes, professors dismiss the idea of a proof, and often claim that because of its 'fundamentalness', the proof requires analytical techniques. That is roughly true, but it is certainly not true that a proof requires complex analysis. Finally, when a proof is given, they often end lamely with a contradiction, not giving any hint as to the construction of a root.

The purpose of this document is to describe a proof of the fundamental theorem of algebra inspired by the proof given in Munkres's *Topology*. Being two dimensional, I represent the argument primarily through diagrams. All results from topology are intuitive and can be expressed layly. Still, I try to give the necessary terminology so that a reader could fill in a precise proof.

2. Fundamental theorem of algebra

Definition 2.1. A path in the complex plane is a continous function $f : [0, 1] \to \mathbb{C}$. If $f(0) = f(1)$ then f is also a loop.

Definition 2.2. Two paths f and g are homotopic (topologically equivalent) if there is some function $H : [0, 1]^2 -> \mathbb{C}$. Such that

$$H(0, t) = f(t) \quad \text{and} \quad H(1, t) = g(t) \tag{2.3}$$

Intuitively, two paths are homotopic if one can be continuously deformed to another.

Lemma 2.4 (From Topology (Winding Number))**.** Take the space $\mathbb{C} - \{a\}$, or the entire complex plane minus the point a. If f and g loops, with f homotopic to $a + e^{2\pi mt}$ and g

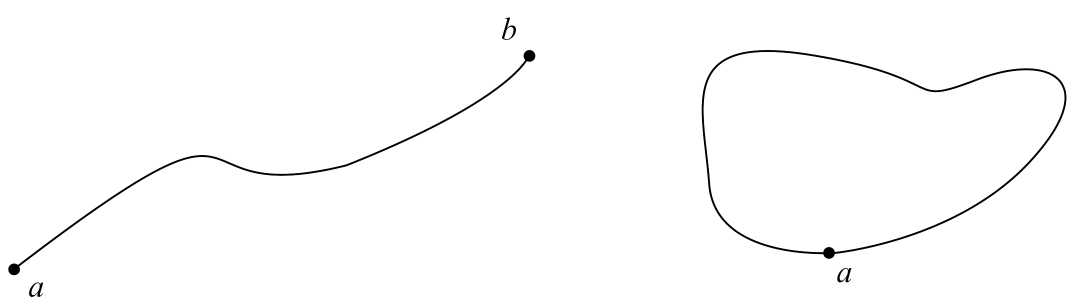


Figure 1. (left) a path and (right) a loop.

homotopic to $a+e^{2\pi nt}$, f and g are homotopic if and only if $m = n$. In other words, if loops are homotopic if they take the same number of trips around a.

All lemma 2.4 says is that any loop is described by the amount of times it turns around a point. Any deformation preserves the number of turns, unless that deformation takes the loop through the point a. I think this is something anyone would expect. This lemma (and the obvious fact that a path is homotopic to itself) are the only things we need for this proof.

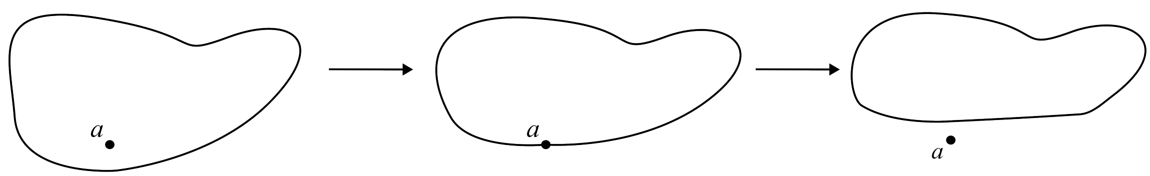


FIGURE 2. A deformation of a loop. a is originally contained within the interior of the loop. In order to reach the exterior, there must be some point along the deformation where a is on the loop.

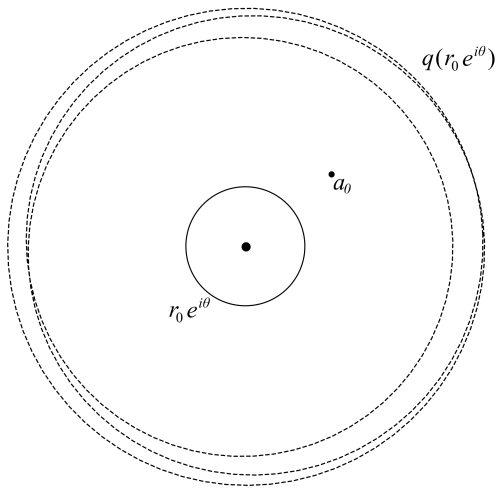
Proof of the fundamental theorem of algebra. Let a polynomial be given by

$$p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z - a_0 \tag{2.5}$$

and we need to assert that there is some z_0 such that $p(z_0) = 0$. For simplicity we will take $q(z) = p(z) + a_0$, and we will search for where $q(z_0) = a_0$. We'll consider z in polar coordinates, $z = re^{i\theta}$. Then,

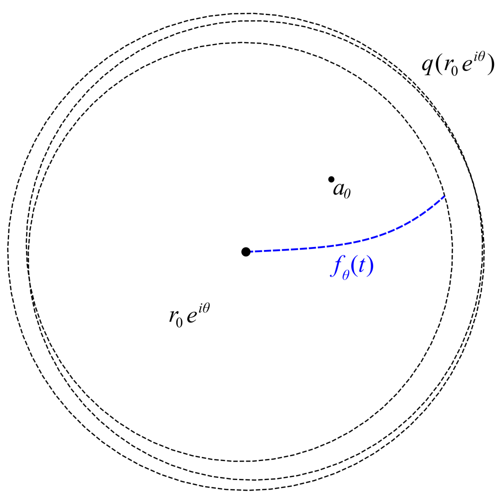
$$q(z) = r^n e^{in\theta} + a_{n-1}r^{n-1}e^{i(n-1)\theta} + ... + a_1 re^{i\theta} \tag{2.6}$$

If we take r fixed and vary θ between 0 and 2π, $q(z)$ describes epicyclic movement. But the first term dominates, so that if we take r_0 large enough, $q(r_0e^{i\theta})$ moves exclusively counter clockwise in the complex plane. Our picture looks as follows:



And we can assume that a_0 is within the interior of $q(r_0e^{i\theta})$ by saying $|a_0| < \min_{0\leq\theta<2\pi} |q(r_0e^{i\theta})|$. Let's consider a path from 0 to a point on $q(r_0e^{i\theta})$ taken now by keeping θ fixed but varying r between 0 and r_0.

$$f_\theta(t) = q(tr_0e^{i\theta}) \tag{2.7}$$



Already, the reasoning is obvious. As we vary θ, f_θ moves counterclockwise with the outer end along $q(r_0e^{i\theta})$. After $\theta = 2\pi$ $f_2\pi$=f_0, and at somewhere along the way f must have crossed a_0.

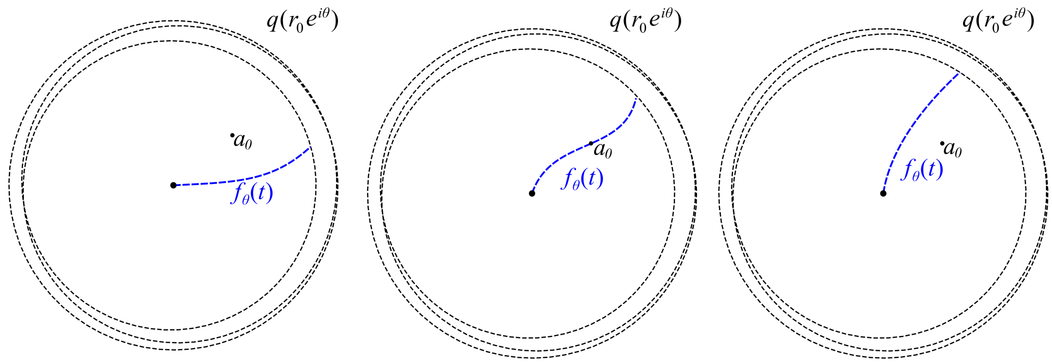


FIGURE 3. The path $f_\theta(t)$ moving through different θ values. In the middle figure, a_0 is achieved for some value of t.

Let's make this more formal. Let's define a path g_θ which first follows f_θ, then travels back along $q(r_0e^{i\theta})$ until $\theta = 0$, then returns to 0 along a straight path.

$$g_\theta(t) = \begin{cases} q(3tr_0t^{i\theta}) & 0 \le t \le \frac{1}{3} \\ q(r_0t^{i\theta(2-3t)}) & \frac{1}{3} < t \le \frac{2}{3} \\ q(r_0(3-3t)) & \frac{2}{3} < t \le 1 \end{cases} \tag{2.8}$$

We consider homotopies on the space $\mathbb{C} - \{a_0\}$ and ask which paths g_θ are homotopic to g_0. Specifically, is it possible that $g_{2\pi}$ is homotopic to g_0? Assume (for contradiction) that $q(z)$ does not ever equal a_0. Then Any two g_{θ_1}, g_{θ_2} are homotopic through the following homotopy.

$$H_{\theta_1,\theta_2}(s,t) = \begin{cases} q(3tr_0e^{i(\theta_1(1-s)+\theta_2(s))}) & 0 \le t \le \frac{1}{3} \\ q(r_0t^{i(\theta_1(1-s)+\theta_2(s))(2-3t)}) & \frac{1}{3} < t \le \frac{2}{3} \\ q(r_0(3-3t)) & \frac{2}{3} < t \le 1 \end{cases} = g_{(\theta_1(1-s)+\theta_2(s))}(t) \tag{2.9}$$

Which has $H_{\theta_1,\theta_2}(0,t) = g_{\theta_1}(t)$ and $H_{\theta_1,\theta_2}(1,t) = g_{\theta_2}(t)$. Because $q(z)$ never equals a_0, we don't have to worry whether the first or third sections intersect a_0.

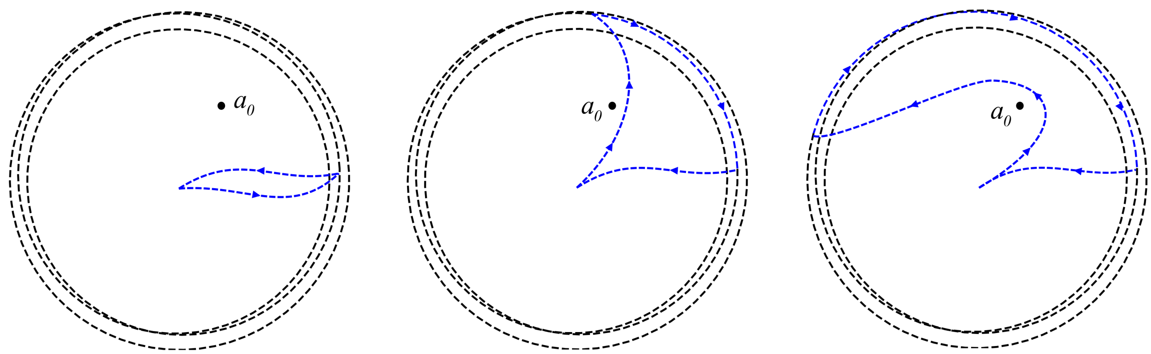


FIGURE 4. Three g_θ for different values of θ. The homotopy H_{θ_1,θ_2} moves through all g_θ for θ between θ_1 and θ_2

By this homotopy, we have that g_0 is homotopic to $g_{2\pi}$. Is that reasonable? The first and third sections of g_0 and $g_{2\pi}$ are identical, but the second section of g_0 is stationary whereas the second section of $g_{2\pi}$ circles a_0 n times. In the below picture I have that g_0 doesn't encircle a_0. That may not be the case, but regardless, if g_0 encircles a_0 k times, $g_{2\pi}$ encircles a_0 $k+n$ times. Thus we reach the contradiction.

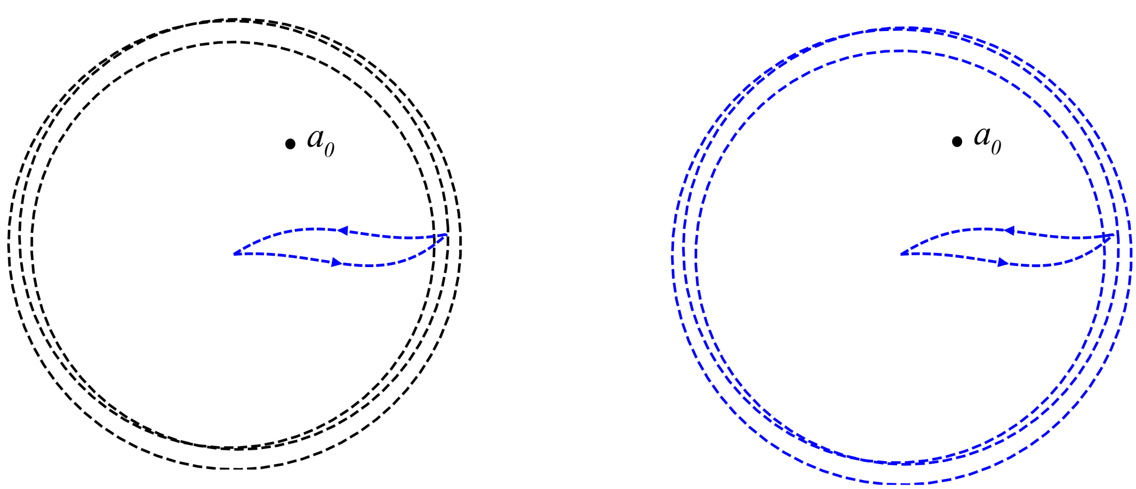


FIGURE 5. (left) g_0 and (right) $g_{2\pi}$. The two loops do not encircle a_0 the same number of times.

Another (slightly more constructive) method of proof could say that between 0 and $\theta^\star$, g_θ does not achieve a_0. Then we show that as $\theta^\star$ approaches 2π, g_θ between 2π and $\theta^\star$ must achieve a_0. $\square$